The submartingale problem for a class of degenerate elliptic operators

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Abstract

We consider the degenerate elliptic operator acting on C^2 functions on $[0,\infty)^d$:

$$\mathcal{L}f(x) = \sum_{i=1}^d a_i(x) x_i^{\alpha_i} \frac{\partial^2 f}{\partial x_i^2}(x) + \sum_{i=1}^d b_i(x) \frac{\partial f}{\partial x_i}(x),$$

where the a_i are continuous functions that are bounded above and below by positive constants, the b_i are bounded and measurable, and the $\alpha_i \in (0,1)$. We impose Neumann boundary conditions on the boundary of $[0,\infty)^d$. There will not be uniqueness for the submartingale problem corresponding to $\mathcal{L}$. If we consider, however, only those solutions to the submartingale problem for which the process spends 0 time on the boundary, then existence and uniqueness for the submartingale problem for $\mathcal{L}$ holds within this class. Our result is equivalent to establishing weak uniqueness for the system of stochastic differential equations

$$dX_t^i = \sqrt{2a_i(X_t)}(X_t^i)^{\alpha_i/2} dW_t^i + b_i(X_t)\,dt + dL_t^{X^i}, \qquad X_t^i \geq 0,$$

where W_t^i are independent Brownian motions and $L_t^{X_i}$ is a local time at 0 for X^i.

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1 Introduction

We consider the degenerate elliptic operator acting on C^2 functions on $[0,\infty)^d$ defined by

$$\mathcal{L}f(x) = \sum_{i=1}^d a_i(x) x_i^{\alpha_i} \frac{\partial^2 f}{\partial x_i^2}(x) + \sum_{i=1}^d b_i(x) \frac{\partial f}{\partial x_i}(x). \tag{1.1}$$

We assume here that the b_i are bounded, the a_i are continuous and bounded above and below by positive constants, and each $\alpha_i \in (0,1)$. We impose zero Neumann boundary conditions on $\partial(\mathbb{R}_+^d)$, where we write $\mathbb{R}_+ = [0,\infty)$. In this paper we investigate whether there is at most one process corresponding to the operator $\mathcal{L}$.

We formulate this question in terms of a submartingale problem. Let $\Omega = C([0,\infty); \mathbb{R}_+^d)$, the continuous functions from $[0,\infty)$ to $\mathbb{R}_+^d$. Define the canonical process X by $X_t(\omega) = \omega(t)$ and let $\mathcal{F}_t$ be the filtration generated by X. Let $x \in \mathbb{R}_+^d$. We say that a probability measure $\mathbb{P}$ on Ω is a solution to the submartingale problem for $\mathcal{L}$ started at x if $\mathbb{P}(X_0 = x) = 1$ and whenever $f \in C^2(\mathbb{R}_+^d)$ such that for each i we have in addition $\partial f / \partial x_i \geq 0$ on $\{x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d : x_i = 0\}$, then $f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\,ds$ is a submartingale with respect to $\mathbb{P}$.

If Y_t is a one dimensional process on $[0,\infty)$, by local time at 0 of Y we mean a continuous nondecreasing process L_t^Y such that L^Y increases only when Y is at 0. Closely related to the operator $\mathcal{L}$ is the system of equations

$$dX_t^i = \sqrt{2a_i(X_t)}(X_t^i)^{\alpha_i/2} dW_t^i + b_i(X_t)dt + dL_t^{X^i}, \qquad i = 1, \ldots, d, \tag{1.2}$$

where $X_0^i = x_i$, $X_t^i \geq 0$ for all t, L^{X^i} is a local time at 0 for X^i, and the W^i are independent one dimensional Brownian motions started at 0.

We say a weak solution to (1.2) exists if there is a probability $\mathbb{P}$ such that (1.2) holds and the W^i are independent Brownian motions under $\mathbb{P}$. Weak

uniqueness holds if given any two solutions $(X_j, W_j, \mathbb{P}_j)$, $j = 1, 2$, the joint law of (X_1, W_1) under $\mathbb{P}_1$ is equal to the joint low of (X_2, W_2) under $\mathbb{P}_2$.

We have assumed that each α_i is in the interval $(0, 1)$, so in fact uniqueness for the submartingale problem for $\mathcal{L}$ does **not** hold. This can be seen even in one dimension: if one looks at the one-dimensional diffusion on natural scale with speed measure $m(dx) = x^{-\alpha}$ for x positive and m equal to 0 on $(-\infty, 0)$, one can put an atom of arbitrary finite mass at 0 and obtain different processes.

If, however, one restricts attention to those solutions to the submartingale problem $\mathcal{L}$ for which the process spends zero time at the boundary, then uniqueness of the submartingale problem does hold. Our main theorem is the following. Let $\Delta = \partial(\mathbb{R}^d_+)$.

Theorem 1.1 *Let* $x \in \mathbb{R}^d_+$

(a) There exists one and only one solution to the submartingale problem for $\mathcal{L}$ *started at* x *that spends zero time in* Δ*, i.e.,*

$$\int_0^\infty 1_\Delta(X_s)ds = 0, \qquad \mathbb{P} - a.s.$$

(b) A weak solution to (1.2) exists that spends zero time in Δ*. Weak uniqueness holds if we restrict attention to those weak solutions that spend zero time in* Δ*.*

Our paper continues the study of degenerate diffusions in the positive orthant begun in [1] and [5]. Those papers concerned the operator $\mathcal{L}$ where all the α_i were equal to 1. In some sense, when the $\alpha_i = 1$ is the critical case, in that then the exact values of the drift coefficients b_i make a large difference to the behavior of the resulting process. When $\alpha_i > 1$, then either the process never attains the boundary, or if it starts on the boundary, never leaves, so the problem then becomes a lower dimensional one. This paper deals with the remaining case.

Although the values of the drift coefficients play less of a role, the results of this paper are not a subset of those in [1]. In fact, they could not be, because here we need the additional assumption that the process spends zero time

on the boundary in order to have uniqueness, while no such assumption is needed in [1].

If $\alpha_i < 1/4$, one can check that a Girsanov transformation allows one to assume that the corresponding b_i can be taken to zero. We wanted to allow the full range of α_i and drift coefficients, so we did not restrict the values of the α_i to $(0, 1/4)$.

As is often the case, uniqueness for a martingale or submartingale problem is often related to the existence of a solution to a PDE problem. That is also the case here, but we do not pursue this connection. Our techniques could also be applied to diffusions on $\mathbb{R}^d$ whose coefficients decay near the i^{th} axis like $|x_i|^{\alpha_i}$.

Our methods differ substantially from those in [1]. That paper used L^2 estimates. Here we prove an analogue of Krylov's inequality and use Littlewood-Paley theory to obtain L^p estimates. These are of independent interest, and could be used to simplify the proof of [1]. In particular, our method of utilizing Littlewood-Paley theory potentially has applications to many other types of martingale problems. (The paper [5] uses C^α estimates and is quite different, both in results and in methods.)

The papers [16], [11], and [6] also consider diffusions with reflection; the latter two consider pathwise uniqueness. Although some smoothness of the domain is needed in these papers, the key difference is that degeneracies of the type given in (1.1) and (1.2) are not allowed.

After a section on preliminaries, we prove an inequality of Krylov type in Section 3. Section 4 concerns existence of solutions. Sections 5 and 6 give the required estimates for the first and second order terms, respectively. The proof of Theorem 1.1 is completed in Section 7. Some of the calculations needed in Section 5 are deferred to Section 8, which is an appendix.

2 Preliminaries

We often write f_i and f_{ii} for the first and second partial derivatives of f with respect to x_i. The Lebsegue measure of a Borel set B will be written $|B|$. We use $\mathbb{R}_+ = [0, \infty)$ and for our state space we will use $\mathbb{R}_+^d$. We often use Δ for the boundary of $\mathbb{R}_+^d$. More precisely, we set $\Delta_i = \{x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d :$

$x_i = 0\}$ and $\Delta = \cup_{i=1}^d \Delta_i$. We use the letter c with or without subscripts to denote finite positive constants whose values are unimportant and may change from place to place.

The connection between Theorem 1.1 (a) and (b) is the following.

Proposition 2.1 *There exists a solution to the submartingale problem for $\mathcal{L}$ started at x if and only if there exists a weak solution to (1.2). The solution to the submartingale problem will be unique if and only if there is weak uniqueness to (1.2). These assertions continue to hold if we restrict attention to probability measures $\mathbb{P}$ such that X_t spends zero time in Δ, $\mathbb{P}$-a.s.*

Proof. The proof of this proposition is very similar to the nondegenerate diffusion case (see [4], Theorem V.1.1) and we give only a brief sketch. If $\mathbb{P}$ is a weak solution to (1.2), then an application of Ito's formula shows that $\mathbb{P}$ will be a solution to the submartingale problem. If $\mathbb{P}$ is a solution to the submartingale problem and we take $f(x) = x_i$, then by the definition of submartingale problem,

$$f(X_t) - \int_0^t \mathcal{L}f(X_s)ds = X_t^i - \int_0^t b_i(X_s)ds$$

is a submartingale, and so by the Doob-Meyer decomposition can be written as a martingale M_t^i plus an increasing process L_t^i. Similarly to the nondegenerate diffusion case, one can show that $X_t^i - \int_0^t b_i(X_s)\, ds$ is a martingale away from the boundary with quadratic variation $\int_0^t a_i(X_s)(X_s^i)^2 ds$. This implies that L_t^i increases only when X_t^i is at 0, and hence is a local time at 0 for X^i. We then proceed as in the proof of the nondegenerate case. $\square$

Any weak solution to (1.2) satisfies a tightness estimate. By this we mean the following.

Proposition 2.2 *If $M > 0, \delta > 0, \eta > 0$, and $t_0 > 0$, there exists $\varepsilon > 0$ such that if $(X, W, \mathbb{P})$ is any weak solution to (1.2) and $S_M = \inf\{t : X_t \notin [0, M]^d\}$, then*

$$\mathbb{P}(\sup_{s,t \le S_M \wedge t_0, |t-s| < \varepsilon} |X_t - X_s| > \delta) < \eta.$$

Proof. It suffices to consider each component of X separately. Fix i. If $s<t$ and $X^i_s>\delta/4$, then by standard estimates we can find ε such that

$$\mathbb{P}(\sup_{s\le t\le s+\varepsilon\le t_0+\varepsilon}|X^i_t-X^i_s|>\delta/8)<\eta; \tag{2.1}$$

this is because L^{X^i} increases only when X^i is at 0. If $X^i_s<\delta/4$, then in order for X^i to be greater than δ within time ε, there must be times $s'<t'$ with $s<s'<t'<s+\varepsilon$ where $X^i_{s'}=\delta/2$ and $X^i_{t'}=3\delta/4$. But by (2.1), the probability of this can be made small by taking ε small. □

When it comes to proving uniqueness of the submartingale problem, it suffices to consider only solutions defined on the canonical probability space $C([0,\infty);\mathbb{R}^d_+)$. If S is a stopping time, we let $\mathbb{Q}_S$ be a regular conditional probability $\mathbb{P}(\,\cdot\circ\theta_S\mid\mathcal{F}_S)$, where θ_S is the usual shift operator of Markov process theory. We denote the corresponding expectation by $\mathbb{E}_{\mathbb{Q}_S}$. Just as in the nondegenerate case, it is easy to see that $\mathbb{Q}_S$ will be a solution to the submartingale problem started at X_S; cf. [4], Proposition VI.2.1.

3 Occupation time estimates

Let

$$\Delta_i=\{x\in\mathbb{R}^d_+:x_i=0\},\qquad \Delta=\cup_{i=1}^d\Delta_i. \tag{3.1}$$

For any process Z and any Borel set C we let $T_C(Z)=T_C=\inf\{t:Z_t\in C\}$ and $\tau_C(Z)=\tau_C=\inf\{t:Z_t\notin C\}$. When C is a single point $\{y\}$, we write instead T_y and τ_y.

We start with an estimate on how long the solution to a one dimensional SDE can spend near 0.

Theorem 3.1 *Suppose $x_0\in[0,\infty)$, W_t is a Brownian motion, a_t and b_t are adapted processes, $c_1^{-1}\le a_t\le c_1$ a.s. for each t, and $|b_t|\le c_1$ a.s. for each t. Suppose either (a) X solves*

$$dX_t=a_t\,X_t^{\alpha/2}dW_t+b_t\,dt+dL^X_t,\qquad X_t\ge 0,\quad X_0=x_0, \tag{3.2}$$

and X spends zero time at 0

or (b) for some $\varepsilon > 0$

$$dX_t = a_t\,(X_t+\varepsilon)^{\alpha/2} dW_t + b_t\,dt + dL_t^X, \qquad X_t \geq 0, \quad X_0 = x_0, \qquad (3.3)$$

where L_t^X is a continuous nondecreasing process that increases only when X_t is at 0. Let $K > 0$. There exists c_2 depending only on K and c_1 such that

$$\mathbb{E}\int_0^{T_K(X)} 1_{[0,\eta]}(X_s)ds \leq c_2\eta^{1-\alpha}.$$

Proof. By first performing a nondegenerate time change, we may without loss of generality suppose that $a_t \equiv 1$. In case (b), Girsanov's theorem and the fact that the diffusion coefficient is bounded below away from 0 tells us that the solution to (3.3) will spend zero time at 0. So we can consider both cases at once if we let $\varepsilon \geq 0$ and specify that X_t spends 0 time at 0. Let $Y_t = X_t^2$. By Ito's formula,

$$\begin{aligned}
dY_t &= 2X_t\,dX_t + d\langle X\rangle_t \\
&= 2X_t(X_t+\varepsilon)^{\alpha/2}dW_t + (X_t+\varepsilon)^{\alpha}dt + 2X_t(X_t+\varepsilon)^{\alpha/2}b_t\,dt + 2X_t\,dL_t^X \\
&= 2Y_t^{1/2}(Y_t^{1/2}+\varepsilon)^{\alpha/2}dW_t + (Y_t^{1/2}+\varepsilon)^{\alpha}dt \\
&\qquad + 2Y_t^{1/2}(Y_t^{1/2}+\varepsilon)^{\alpha/2}b_t\,dt,
\end{aligned}$$

where we use the fact that $X_t\,dL_t^X$ is 0 since L_t^X only increases when X_t is at 0.

Note Y_t is always nonnegative. By [7], Section 6.1, if $Z_0 = Y_0$ and Z_t solves

$$dZ_t = 2Z_t^{1/2}(Z_t^{1/2}+\varepsilon)^{\alpha/2}dW_t + (Z_t^{1/2}+\varepsilon)^{\alpha}dt - 2c_1Z_t^{1/2}(Z_t^{1/2}+\varepsilon)^{\alpha/2}dt,$$

with $Z_t \geq 0$ for all t, then $Z_t \leq Y_t$ for all t.

Let U_t be the continuous strong Markov process corresponding to the operator

$$\tfrac{1}{2}(x+\varepsilon)^{\alpha}f''(x) - c_1f'(x)$$

with reflection at 0; in the case $\varepsilon = 0$ we specify that the process spends 0 time at 0. Then for some Brownian motion $\widehat{W}$, U will be a weak solution to

$$dU_t = (U_t+\varepsilon)^{\alpha/2}d\widehat{W}_t - c_1\mathrm{sgn}\,(U_t)\,dt + dL_t^U,$$

where L_t^U is a continuous nondecreasing process that increases only when U_t is at 0. Let $V_t = U_t^2$. A calculation using Ito's formula as above shows that V_t solves the same equation as Z_t except with a different Brownian motion. Note that the equation defining Z_t locally satisfies the conditions of [10], so we have pathwise uniqueness, hence uniqueness in law, and thus the law of Z and V are the same. Therefore

$$\begin{aligned}
\mathbb{E}\int_0^{T_K(X)} 1_{[0,\eta]}(X_s)ds &= \mathbb{E}\int_0^{T_{K^2}(Y)} 1_{[0,\eta^2]}(Y_s)ds \\
&\le \mathbb{E}\int_0^{T_{K^2}(Z)} 1_{[0,\eta^2]}(Z_s)ds \\
&= \mathbb{E}\int_0^{T_{K^2}(V)} 1_{[0,\eta^2]}(V_s)ds \\
&= \mathbb{E}\int_0^{T_K(U)} 1_{[0,\eta]}(U_s)ds.
\end{aligned}$$

We compute the scale function $s(x)$ for the process U_t and find that it is determined by

$$\log s'(x) = \int_0^x \frac{2c_1}{(y+\varepsilon)^\alpha}dy.$$

It follows that $s(U_t)$ corresponds to the operator

$$A_\varepsilon(x)(x+\varepsilon)^\alpha f''(x)$$

where $A_\varepsilon(x)$ is a function of x satisfying

$$c_3 \le A_\varepsilon(x) \le c_4, \qquad |x| \le K,$$

and $0 < c_3 < c_4 < \infty$ do not depend on ε; furthermore the speed measure for the process has no atom at 0. Moreover, we see that $s(\eta)/\eta$ is bounded by c_5 for η small. So

$$\mathbb{E}\int_0^{T_K(U)} 1_{[0,\eta]}(U_r)dr \le \mathbb{E}\int_0^{T_{s(K)}(s(U))} 1_{[0,s(\eta)]}(s(U_r))dr.$$

On the other hand, the term on the right is bounded by (see [4], Section IV.3)

$$c_7\int_0^{c_6} 1_{[0,c_5\eta]}(x)\frac{1}{A_\varepsilon(x)(x+\varepsilon)^\alpha}dx \le c_8\int_0^{c_5\eta}\frac{1}{x^\alpha}dx,$$

and this in turn is bounded by

$$c_9\eta^{1-\alpha},$$

independently of ε. This proves the theorem. $\square$

Corollary 3.2 *Suppose X_t satisfies the hypotheses of Theorem 3.1. Then there exist positive c_1 and c_2 such that for any $\gamma \le K$, the probability of more than m upcrossings of $[0,\gamma]$ by X_t before time T_K is bounded by*

$$c_1(1-c_2\gamma)^m.$$

Proof. For any process R on $\mathbb{R}$, let $T_0^R = 0$, $S_i^R = \inf\{t > T_{i-1}^R : R_t = 0\}$, $T_i^R = \{t > S_i^R : R_t = \gamma\}$, $i \ge 1$. Let U be the process defined in the proof of Theorem 3.1. Since $s(U)$ is on natural scale,

$$\begin{aligned}\mathbb{P}^\gamma(S_1^U < T_K(U)) &= \mathbb{P}^{s(\gamma)}(S_1^{s(U)} < T_{s(K)}(s(U)) = 1 - \frac{s(\gamma)}{s(K)}\\ &\le 1 - c_3\gamma.\end{aligned}$$

As in the proof of Theorem 3.1, X_t is stochastically larger than the process U_t. Therefore if $X_0 = \gamma$,

$$\mathbb{P}(S_1^X < T_K(X)) \le \mathbb{P}^\gamma(S_1^U < T_K(U)) \le 1 - c_3\gamma.$$

Recall we use $\mathbb{Q}_S$ for a regular conditional probability for $\mathbb{P}(\,\cdot \circ \theta_S \mid \mathcal{F}_S)$. Under $\mathbb{Q}_{T_i^X}$ the process X_t satisfies the hypotheses of Theorem 3.1, and so

$$\begin{aligned}\mathbb{P}(T_{i+1}^X \le T_K(X)) &\le \mathbb{E}[\mathbb{Q}_{T_i^X}(S_1^X < T_K(X)); T_i^X \le T_K(X)]\\ &\le (1-c_3\gamma)\mathbb{P}(T_i^X \le T_K(X)).\end{aligned}$$

By induction,

$$\mathbb{P}(T_i^X \le T_K(X)) \le c_4(1-c_3\gamma)^i,$$

which implies our result. $\square$

Theorem 3.3 *Let $M > 1$ be fixed and $\lambda > 0$. There exist p_0 and c_1 depending on $\alpha_1, \ldots, \alpha_d, M, \lambda$, and d such that if $f \in L^{p_0}(\mathbb{R}^d)$ and f is supported in $[0, M]^d$, then*

$$\Big|\mathbb{E}\int_0^\infty e^{-\lambda s} f(X_s)\, ds\Big| \le c_1 \|f\|_{p_0} \tag{3.4}$$

for any solution to (1.2).

Proof. Let us set $D = [0, 2M]^d$ and $E = [0, M]^d$. Let $A \subset E$ and let $\varepsilon = |A|$. Let $\delta > 0$ be a small positive real to be chosen later and let $F = [\varepsilon^\delta, 2M]^d$.

Our first goal is to show that there exists K_1 and γ_1 not depending on ε such that if $A \subset F$, then

$$\mathbb{E}\int_0^{\tau_F} 1_A(X_s)\, ds \le c_2 \varepsilon^{-\delta K_1 + \gamma_1}. \tag{3.5}$$

Define

$$Y_t^i = [1 - \tfrac{\alpha_i}{2}]^{-1} (X_t^i)^{1-(\alpha_i/2)}.$$

We use Ito's formula to see that for $t \le \tau_F$

$$dY_t^i = \sqrt{2} a_i(X_t)\, dW_t^i + \Big[(Y_t^i)^{-\alpha_i/(2-\alpha_i)} b_i(X_t) - \frac{\alpha_i}{4 Y_t^i}\Big]\, dt.$$

Since we are on the set F, there is no issue of the degeneracy of X_t^i at 0 causing problems. Notice that on F the drift coefficient of Y^i is bounded by $c_3 \varepsilon^{-K_2 \delta}$. Define the map $\Gamma : F \to \mathbb{R}_+^d$ by

$$\Gamma(x_1, \ldots, x_d) = ([1 - \tfrac{\alpha_1}{2}]^{-1} (x_1)^{1-(\alpha_1/2)}, \ldots, [1 - \tfrac{\alpha_d}{2}]^{-1} (x_d)^{1-(\alpha_d/2)}).$$

To obtain (3.5) it suffices to bound

$$\mathbb{E}\int_0^{\tau_{\Gamma(F)}} 1_{\Gamma(A)}(Y_s)\, ds. \tag{3.6}$$

Note

$$|\Gamma(A)| \le c_4 \prod_{i=1}^d (\varepsilon^\delta)^{-\alpha_i/2} |A| \le c_5 \varepsilon^{-\delta K_3 + 1}$$

for some $K_3 > 0$. Let $\overline{Y}$ be the process whose coefficients agree with those of Y when the process is in F and satisfy $d\overline{Y}_t^i = \sqrt{2} |\overline{Y}_t^i|^{\alpha_i/2}\, dW_t^i$ outside of F.

Let G be a ball of radius $c_6 M$ such that contains F. If we look at the first component of $\overline{Y}$, the time for $|\overline{Y}_t^1|$ to exceed $2(c_6+1)M$ is less than or equal to the time $|\overline{Y}_t^1|$ spends in $[0, 2(c_6+1)M]$ before exceeding $4(c_6+1)M$, and this latter amount of time has finite expectation by Theorem 3.1. Therefore $\mathbb{E}\tau_G(\overline{Y})$ is bounded by a constant c_7. We now use Krylov [8] to obtain the bound

$$\mathbb{E}\int_0^{\tau_G(\overline{Y})} 1_{\Gamma(A)}(\overline{Y}_s)\, ds \le c_8(1 + c_9\varepsilon^{-K_2\delta}\mathbb{E}\tau_G(\overline{Y}))|\Gamma(A)|^{1/d}.$$

This inequality follows from a passage to the limit in equation (4) of [8]. We therefore have

$$\begin{aligned}\mathbb{E}\int_0^{\tau_F(X)} 1_A(X_s)\, ds &\le \mathbb{E}\int_0^{\tau_{\Gamma(F)}(Y)} 1_{\Gamma(A)}(Y_s)\, ds \\ &\le \mathbb{E}\int_0^{\tau_G(\overline{Y})} 1_{\Gamma(A)}(\overline{Y}_s)\, ds \\ &\le c_{10}(1 + \varepsilon^{-K_2\delta})\varepsilon^{(-\delta K_3+1)/d}.\end{aligned} \tag{3.7}$$

This proves (3.5).

Next we will show that if $A \subset E$, then there exists K_4, K_5, and γ_2 such that

$$\mathbb{E}\int_0^{\tau_E} 1_A(X_s)\, ds \le c_{11}(\varepsilon^{-K_4\delta+\gamma_2} + \varepsilon^{\delta K_5}). \tag{3.8}$$

Write $A = A_1 \cup A_2$, where $A_1 = A \cap (\cup_{i=1}^d \{0 \le x_i \le \varepsilon^\delta\})$ and $A_2 = A \setminus A_1$. By Theorem 3.1, we know

$$\mathbb{E}\int_0^{\tau_D} 1_{A_1}(X_s)\, ds \le \sum_{i=1}^d \mathbb{E}\int_0^{\tau_D} 1_{[0,\varepsilon^\delta]}(X_s^i)\, ds \le c_{12}\varepsilon^{\delta K_4}. \tag{3.9}$$

So we need to bound

$$\mathbb{E}\int_0^{\tau_D} 1_{A_2}(X_s)\, ds.$$

Let $T_0 = 0$, $S_i = \inf\{t > T_{i-1} : X_t \in F\}$, and $T_i = \inf\{t > S_i : X_t \notin [\varepsilon^\delta/2, 2M]^d\}$, $i \ge 1$. Recall that $\mathbb{Q}_{S_i}$ is used for a regular conditional proba-

bility. Then

$$\begin{aligned}\mathbb{E}\int_0^{\tau_E} 1_{A_2}(X_s)\,ds &= \sum_{i=1}^{\infty} \mathbb{E}\Big[\int_{S_i}^{T_i} 1_{A_2}(X_s); ds; S_i < \tau_E\Big]\\ &= \sum_{i=1}^{\infty} \mathbb{E}\Big[\mathbb{E}_{\mathbb{Q}_{S_i}} \int_0^{T_1} 1_{A_2}(X_s)\,ds; S_i < \tau_E\Big]\\ &\le c_{13}\varepsilon^{-K_1\delta+\gamma_1} \sum_{i=1}^{\infty} \mathbb{P}(S_i < \tau_E).\end{aligned}$$

Now in order for S_i to be less than τ_E, we must have for some $j \le d$ at least $(i-1)/d$ upcrossings of X^j from $\varepsilon^\delta/2$ to ε^δ before hitting the level M. We know by Corollary 3.2 that the probability of this happening is less than $c_{14}(1 - c_{15}\varepsilon^\delta)^{(i-1)/d}$. Therefore

$$\sum_{i=1}^{\infty} \mathbb{P}(S_i < \tau_E) \le c_{16}\varepsilon^{-K_5\delta}.$$

Combining with (3.9), we have

$$\mathbb{E}\int_0^{\tau_E} 1_A(X_s)\,ds \le c_{17}(\varepsilon^{\delta K_4} + \varepsilon^{-K_1\delta+\gamma_1-K_5\delta}), \tag{3.10}$$

which yields (3.8).

The next step is to show that there exists K_6 and γ_3 such that if $A \subset E$, then

$$\mathbb{E}\int_0^{\infty} e^{-\lambda s} 1_A(X_s)\,ds \le c_{18}(\varepsilon^{-K_6\delta+\gamma_3} + \varepsilon^{\delta K_4}). \tag{3.11}$$

Let $V_0 = 0$, $U_i = \inf\{t > V_{i-1} : X_t \in E\}$, $V_i = \inf\{t > U_i : X_t \notin D\}$, $i \ge 1$. We have for $x \notin [0, 2M)^d$ that there exists c_{19} and c_{20} such that $\mathbb{P}^x(U_1 > c_{19}) > c_{20}$. This is because if $x \notin [0, 2M)^d$, then at least one coordinate of X is greater than or equal to $2M$, and in this range this coordinate is a diffusion whose diffusion coefficients are bounded above and below and whose drift coefficient is bounded above; therefore it cannot move a distance M too quickly. We conclude

$$\mathbb{E}e^{-\lambda U_1} \le (1 - c_{20}) + c_{20}e^{-\lambda c_{19}} := \rho.$$

Note that $\rho < 1$. We then have

$$\begin{aligned}\mathbb{E}e^{-\lambda U_i} &= \mathbb{E}\Big[e^{-\lambda V_{i-1}}\mathbb{E}\Big[e^{-\lambda U_i} \mid \mathcal{F}_{V_{i-1}}\Big]\Big] \\ &\le \mathbb{E}\Big[e^{-\lambda U_{i-1}}\mathbb{E}_{\mathbb{Q}_{V_{i-1}}} e^{-\lambda U_1}\Big] \le \rho\mathbb{E}e^{-\lambda U_{i-1}}.\end{aligned}$$

So by induction

$$\mathbb{E}e^{-\lambda U_i} \le \rho^{i-1}.$$

Then

$$\begin{aligned}\mathbb{E}\int_0^\infty e^{-\lambda s}1_A(X_s)\,ds &= \sum_{i=1}^\infty \mathbb{E}\int_{U_i}^{V_i} e^{-\lambda s}1_A(X_s)\,ds \\ &= \sum_{i=1}^\infty e^{-\lambda U_i}\mathbb{E}_{\mathbb{Q}_{U_i}}\int_0^{V_1} e^{-\lambda s}1_A(X_s)\,ds \\ &\le c_{21}(\varepsilon^{-K_4\delta+\gamma_2} + \varepsilon^{K_4\delta})\sum_{i=1}^\infty \mathbb{E}e^{-\lambda U_i} \\ &\le c_{22}(\varepsilon^{-K_4\delta+\gamma_2} + \varepsilon^{K_4\delta}),\end{aligned}$$

which proves (3.11).

We now choose $\delta = \gamma_3/(2K_6)$ and we obtain

$$\mathbb{E}\int_0^\infty e^{-\lambda s}1_A(X_s)\,ds \le c_{23}\varepsilon^{\gamma_4}. \tag{3.12}$$

Let $p_0 = 2/\gamma_4$. To complete the proof, suppose $f \in L^{p_0}(\mathbb{R}^d_+)$ with support in $[0,M]^d$. By multiplying by a constant, it suffices to consider the case where $\|f\|_{p_0} = 1$. Without loss of generality we may also suppose $f \ge 0$. Let $A_n = \{x : f(x) \ge 2^n\}$. Then

$$|A_n| \le \frac{\|f\|_{p_0}^{p_0}}{(2^n)^{p_0}} = 2^{-np_0}.$$

We then have

$$\mathbb{E}\int_0^\infty 1_{A_n}(X_s)\,ds \le c_{24}2^{-np_0\gamma_4}.$$

Thus

$$\mathbb{E}\int_0^\infty e^{-\lambda s} f(X_s)\,ds \le 1 + \sum_{n=0}^\infty 2^{n+1}\mathbb{E}\int_0^\infty e^{-\lambda s} 1_{A_n}(X_s)\,ds$$
$$\le 1 + \sum_{n=0}^\infty 2^{n+1} c_{24} 2^{-np_0\gamma_4} \le c_{25} < \infty.$$

Since $\|f\|_{p_0} = 1$, the proof is complete. □

4 Existence

In this section we prove existence of a solution to the submartingale problem. for the operator $\mathcal{L}$ defined in (1.1) with Neumann boundary conditions on Δ.

There are two complications that are not present in the usual case: we need to show that our solution spends zero time on the set Δ defined in (3.1); and unless the α_i are small, we cannot use the Girsanov transformation to reduce to the case of zero drift.

Let $\mathcal{L}^\varepsilon$ be the operator defined by

$$\mathcal{L}^\varepsilon f(x) = \sum_{i=1}^d a_i(x)(x_i+\varepsilon)^{\alpha_i} f_{ii}(x) + \sum_{i=1}^d b_i(x) f_i(x),$$

again with reflecting boundary conditions on $\Delta = \partial(\mathbb{R}^d_+)$. The diffusion coefficients are uniformly positive definite and continuous and are of at most linear growth, so there exists a unique solution to the submartingale problem for $\mathcal{L}^\varepsilon$ started from x_0 for every x_0; let us denote it $\mathbb{P}_\varepsilon$. (We reflect the coefficients over the coordinate axes, construct the solution to the corresponding martingale problem on $\mathbb{R}^d$, and then look at the law of $(|X^1|, \ldots, |X^d|)$.)

Using Proposition 2.2 it is standard that the $\mathbb{P}_\varepsilon$ are a tight sequence of probability measures on $C([0,\infty);\mathbb{R}^d_+)$ and there must exist a subsequence ε_j such that $\mathbb{P}_{\varepsilon_j}$ converges weakly. Denote the limit measure by $\mathbb{P}$ and the corresponding expectation by $\mathbb{E}$. It is obvious that $\mathbb{P}(X_0 = x_0) = 1$.

Proposition 4.1 *Under $\mathbb{P}$ the process spends zero time in Δ, i.e.,*

$$\int_0^\infty 1_\Delta(X_s)ds = 0, \qquad \mathbb{P} - a.s.$$

Proof. Under $\mathbb{P}_\varepsilon$, the ith component of X_t will satisfy an SDE of the form

$$dX_t^i = \sqrt{2a_i(X_t)}(X_t^i + \varepsilon)^{\alpha_i/2} dW_t^i + b_i(X_t)\, dt + dL_t^{X^i},$$

where L^{X^i} is a local time at 0. Applying Theorem 3.1 the amount of time X_t^i spends in $[0, \eta]$ before exceeding K under $\mathbb{P}_\varepsilon$ is bounded by $c_1 \eta^{1-\alpha_i}$, where c_1 may depend on K but not ε. Taking a limit

$$\mathbb{E}_\mathbb{P} \int_0^{\tau_K(X^i)} 1_{[0,\eta]}(X_s^i)ds \le c_1 \eta^{1-\alpha_i}.$$

Letting $\eta \to 0$, we have

$$\mathbb{E}_\mathbb{P} \int_0^{\tau_K(X^i)} 1_{\Delta_i}(X_s)\, ds = 0.$$

Since K is arbitrary, and using this argument for each $i = 1, \dots, d$, we obtain

$$\mathbb{E}_\mathbb{P} \int_0^\infty 1_\Delta(X_s)ds = 0, \tag{4.1}$$

and hence the amount of time spent in Δ is 0 almost surely. $\square$

Proposition 4.2 $\mathbb{P}$ *is a solution to the submartingale problem for $\mathcal{L}$ started at x_0.*

Proof. To prove that $\mathbb{P}$ is a solution to the submartingale problem for $\mathcal{L}$ started at x_0, we need to prove that $M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds$ is a submartingale for every $f \in C^2$ such that $f_i \ge 0$ on Δ_i. To do that, it suffices to show M_t^f is a submartingale for every such $f \in C^2$ that in addition has compact support. Take such an f. It will suffice to show

$$\mathbb{E}[M_t^f Y] \ge \mathbb{E}[M_s^f Y] \tag{4.2}$$

whenever $s < t$, $n > 0$, $r_1 \leq r_2 \leq \cdots \leq r_n \leq s$, and $Y = \prod_{i=1}^n g_i(X_{r_i})$ where $g_1, \ldots, g_n$ are bounded continuous functions with compact support. We know

$$\mathbb{E}_\varepsilon\Big[\Big\{f(X_t) - f(X_0) - \int_0^t \mathcal{L}^\varepsilon f(X_u)du\Big\}Y\Big] \\ \geq \mathbb{E}_\varepsilon\Big[\Big\{f(X_s) - f(X_0) - \int_0^s \mathcal{L}^\varepsilon f(X_u)du\Big\}Y\Big]$$

since $f(X_t) - f(X_0) - \int_0^t \mathcal{L}^\varepsilon f(X_u)\,du$ is a submartingale under $\mathbb{P}_\varepsilon$. Since $f(X_s)Y$, $f(X_t)Y$, and $f(X_0)Y$ are each continuous functionals of the path, then the expectations under $\mathbb{P}_{\varepsilon_j}$ converge to the expectations under $\mathbb{P}$, and we will be done if we prove

$$\mathbb{E}_{\varepsilon_j}\Big[Y\int_0^t \mathcal{L}^{\varepsilon_j} f(X_u)du\Big] \to \mathbb{E}\Big[Y\int_0^t \mathcal{L} f(X_u)du\Big] \tag{4.3}$$

and also with t replaced by s. We will do only the t case; the s case is almost identical.

Since f is C^2 with compact support, then $\mathcal{L}^\varepsilon f(x) \to \mathcal{L} f(x)$ uniformly. So it suffices to prove

$$\mathbb{E}_{\varepsilon_j}\Big[Y\int_0^t \mathcal{L} f(X_u)du\Big] \to \mathbb{E}\Big[Y\int_0^t \mathcal{L} f(X_u)du\Big]. \tag{4.4}$$

Let $F(x) = \sum_{i=1}^d b_i(x) f_i(x)$. Since the a_i are continuous, then the $\mathbb{P}^{\varepsilon_j}$ expectations of

$$Y\int_0^t \sum_i a_i(X_u)du$$

converge to the expectation under $\mathbb{P}$, and it suffices to prove

$$\mathbb{E}_{\varepsilon_j}\Big[Y\int_0^t F(X_u)du\Big] \to \mathbb{E}\Big[Y\int_0^t F(X_u)du\Big]. \tag{4.5}$$

Let $\delta > 0$. Pick K large so that

$$\mathbb{P}_\varepsilon(T_K < t) < \delta.$$

By tightness, this can be done uniformly in ε and the same inequality holds with $\mathbb{P}$ in place of $\mathbb{P}_\varepsilon$. So since δ is arbitrary, it suffices to show

$$\mathbb{E}_{\varepsilon_j}\Big[Y\int_0^t G(X_u)du\Big] \to \mathbb{E}\Big[Y\int_0^t G(X_u)du\Big], \tag{4.6}$$

where $G(x) = F(x)$ for $|x| \le K$ and 0 otherwise. Let $\gamma > 0$. We can find a continuous bounded function H that is 0 outside of $B(0, K)$, that is equal to G except on a set of Lebesgue measure less than γ, and where $\|H\|_\infty = \|G\|_\infty$. Since H is continuous,

$$\mathbb{E}_{\varepsilon_j}\Big[Y \int_0^t H(X_u)du\Big] \to \mathbb{E}\Big[Y \int_0^t H(X_u)du\Big]. \tag{4.7}$$

But by Theorem 3.3 we have

$$\mathbb{E}_\varepsilon \int_0^{T_K} |H(X_u) - G(X_u)|du \le c_2(K, \gamma),$$

uniformly in ε, which can be made less than δ if we take γ small enough. Passing to the limit along ε_j, we have the same result when $\mathbb{E}_\varepsilon$ is replaced by $\mathbb{E}$. This, (4.7), and the facts that $\mathbb{P}_\varepsilon(T_K < t) < \delta$ and $\mathbb{P}(T_K < t) < \delta$ suffice to establish (4.6). □

5 First order estimates

We first consider the continuous strong Markov process Z_t on $[0, \infty)$ associated with the operator

$$\mathcal{A}_Z f(x) = x^\alpha f''(x).$$

Here $\alpha \in (0, 1)$ and we impose reflecting boundary conditions at 0. More precisely, we have a process on natural scale whose speed measure has no atom at 0 and does not charge $(-\infty, 0]$.

Let

$$b = 1 - \frac{\alpha}{2},$$

and note that $b \in (\frac{1}{2}, 1)$. If we set

$$Y_t = \frac{1}{b\sqrt{2}} Z_t^b,$$

a straightforward calculation shows that Y is a continuous strong Markov process on $[0, \infty)$ associated to the operator

$$\mathcal{A}_Y f(x) = \tfrac{1}{2} f''(x) + \frac{b-1}{2bx} f'(x)$$

with reflection at 0, i.e., a Bessel process of order $\delta = \frac{b-1}{b} + 1$ with reflection at 0. By [15] the transition densities of Y (with respect to Lebesgue measure) are given by

$$p_Y(t,x,y) = \Big(\frac{y}{x}\Big)^{\nu} \frac{y}{t} e^{-(x^2+y^2)/2t} I_\nu(xy/t),$$

where $\nu = \frac{\delta}{2} - 1 = -\frac{1}{2b}$ and I_ν is the standard modified Bessel function.

A change of variables then gives

$$p_Z(t,x,y) = \frac{c_1}{t} y^{2b-\frac{3}{2}} e^{-c_2 y^{2b}/2t} x^{\frac{1}{2}} e^{-c_2 x^{2b}/2t} I_\nu(c_2 x^b y^b/t) \tag{5.1}$$

and we have the scaling relationship

$$p_Z(t,x,y) = t^{-1/2b} p_Z(1, xt^{-1/2b}, yt^{-1/2b}). \tag{5.2}$$

We will need the following lemma, the proof of which is given in the appendix.

Lemma 5.1 *There exists a constant c_1 such that*

$$\sup_x \int_0^\infty \Big| \frac{\partial}{\partial x} p_Z(t,x,y) \Big| dy \le c_1 t^{-\frac{1}{2b}}; \tag{5.3}$$

$$\sup_y y^\alpha \int_0^\infty \Big| \frac{\partial}{\partial x} p_Z(t,x,y) \Big| x^{-\alpha} dx \le c_1 t^{-\frac{1}{2b}}. \tag{5.4}$$

Let P_t be the semigroup for Z_t, i.e., $P_t f(x) = \mathbb{E}^x f(Z_t)$. Let $\mu(dx) = x^{-\alpha} dx$ and we consider the space $L^2(\mathbb{R}_+, \mu)$. We use the above estimates to prove

Proposition 5.2 *Suppose $p \in (1,\infty)$. There exists a constant c_1 depending only on p such that*

$$\|(P_t f)'\|_p \le c_1 t^{-1/2b} \|f\|_p, \qquad f \in L^2(\mathbb{R}_+, \mu).$$

Proof. Fix $t > 0$ and write $K(x,y)$ for $|\frac{\partial}{\partial x} p(t,x,y)|$. Let q be the conjugate

exponent to p. Then by Lemma 5.1 we have

$$\begin{aligned}
\|(P_t f)'\|_p^p &\leq \int_{\mathbb{R}_+} \Big[\int_{\mathbb{R}_+} K(x,y)|f(y)|\,dy \Big]^p x^{-\alpha} dx \\
&= \int_{\mathbb{R}_+} \Big[\int_{\mathbb{R}_+} K(x,y)^{1/q} K(x,y)^{1/p} |f(y)|\,dy \Big]^p x^{-\alpha} dx \\
&\leq \int_{\mathbb{R}_+} \Big[\int_{\mathbb{R}_+} K(x,y)\,dy \Big]^{p/q} \Big[\int_{\mathbb{R}_+} K(x,y)|f(y)|^p\,dy \Big] x^{-\alpha} dx \\
&\leq c_2 t^{-(1/2b)(p/q)} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} K(x,y) x^{-\alpha}\,dx\,|f(y)|^p\,dy \\
&\leq c_3 t^{-(1/2b)((p/q)+1)} \int_{\mathbb{R}_+} |f(y)|^p y^{-\alpha}\,dy.
\end{aligned}$$

Now take p-th roots of both sides. $\square$

Now let us turn to the d-dimensional case. We suppose Z_t^i is the process on $\mathbb{R}$ corresponding to the operator

$$\mathcal{A}_i f(x) = x^{\alpha_i} f''(x),$$

with the speed measure having no atom at 0 and not charging $(-\infty, 0)$ and $\alpha_i \in (0,1)$. Set $b_i = 1 - \frac{\alpha_i}{2}$. We let $p_i(t,x,y)$ denote the transition densities of Z_t^i, P_t^i the corresponding semigroups, let $Z_t = (Z_t^1, \ldots, Z_t^d)$, and let $p(t,x,y) = \prod_{i=1}^d p_i(t, x_i, y_i)$ be the transition densities for Z when $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d)$. Let P_t now denote the semigroup for Z. Let μ_i be the measure on $\mathbb{R}_+$ whose Radon-Nikodym derivative with respect to d-dimensional Lebesgue measure is $x_i^{-\alpha_i}$ and let μ be the measure on $\mathbb{R}_+^d$ given by

$$\mu(dx) = \prod_{i=1}^d \mu_i(dx_i). \tag{5.5}$$

We have the analogue of Proposition 5.2.

Proposition 5.3 *There exists a constant c_1 such that for each i*

$$\Big\| \frac{\partial (P_t f)}{\partial x_i} \Big\|_p \leq c_1 t^{-1/2b_i} \|f\|_p, \qquad f \in L^p(\mathbb{R}_+^d, \mu).$$

Proof. We will prove this in the case $i = 1$, the case for other i's being exactly similar. Let $\overline{x} = (x_2, \ldots, x_d)$. Let $f \in L^2(\mathbb{R}^d_+, \mu)$ and set

$$F(x_1; \overline{x}) = \int \cdots \int \prod_{j=2}^{d} p_j(t, x_j, y_j) f(y_1, y_2, \ldots y_d)\, dy_2 \cdots dy_d.$$

Then

$$\Big| \frac{\partial}{\partial x_1} P_t f(x) \Big| = \Big| \frac{\partial}{\partial x_1} P^1_t F(x_1; \overline{x}) \Big|,$$

and so by Proposition 5.2 we have

$$\int \Big| \frac{\partial P_t f(x)}{\partial x_1} \Big|^p \mu_1(dx_1) \le c_2 t^{-p/2b_1} \int |F(x_1; \overline{x})|^p \mu_1(dx_1).$$

If we integrate both sides with respect to $\overline{\mu}(dx_2 \cdots dx_d) = \prod_{j=2}^d \mu_j(dx_j)$, we will have our result provided we show

$$\int |F(x_1; \overline{x})|^p \mu(dx) \le \int |f|^p \mu(dx). \tag{5.6}$$

To prove (5.6) let $\overline{P}_t$ be the semigroup corresponding to $(Z^2_t, \ldots, Z^d_t)$. It is easy to check that $\sum_{j=2}^d \mathcal{A}_j$ is self-adjoint with respect to the measure $\overline{\mu}$. Therefore, using Jensen's inequality,

$$\|\overline{P}_t g\|_{L^p(\overline{\mu})} \le \|g\|_{L^p(\overline{\mu})}, \qquad g : \mathbb{R}^{d-1}_+ \to \mathbb{R},$$

or

$$\int |\overline{P}_t g(x)|^p \overline{\mu}(dx) \le \int |g(x)|^p \overline{\mu}(dx). \tag{5.7}$$

We hold x_1 fixed and apply this to $g(\overline{x}; x_1) = f(x_1, \ldots, x_d)$. Note $\overline{P}_t g(\cdot; x_1) = F(x_1; \overline{x})$. So applying (5.7) to this g, we have

$$\int |F(x_1; \overline{x})|^p \overline{\mu}(dx) \le \int |f(x_1, \ldots, x_d)|^p \overline{\mu}(dx).$$

(5.6) follows by integrating both sides of this equation with respect to $\mu_1(dx_1)$. $\square$

Our main result of this section is the following. Let

$$R_\lambda f = \int_0^\infty e^{-\lambda t} P_t f\, dt.$$

Theorem 5.4 *There exists c_1 such that for each i*

$$\|\partial(R_\lambda f)/\partial x_i\|_p \le c_1 \lambda^{\frac{1}{2b_i}-1} \|f\|_p.$$

Proof. Since $-1/(2b_i) > -1$, the result follows from Proposition 5.3, dominated convergence, and Minkowski's inequality for integrals:

$$\begin{aligned}\Big\|\frac{\partial}{\partial x_i}\int_0^\infty e^{-\lambda t} P_t f\,dt\Big\|_p &= \Big\|\int_0^\infty e^{-\lambda t}\frac{\partial P_t f}{\partial x_i} dt\Big\|_p\\ &\le \int_0^\infty e^{-\lambda t} c_2 t^{-1/(2b_i)} dt \|f\|_p.\end{aligned}$$

□

Remark 5.5 Only very minor changes are needed to get the same conclusion as in Theorem 5.4 if we instead set R_λ to be the resolvent for the operator $\sum_{i=1}^d a_i \mathcal{A}_i$, where the a_i are strictly positive finite constants.

6 Second order estimates

Let

$$\mathcal{A}_i f(x) = |x_i|^{\alpha_i} f_{ii}(x), \qquad x \in \mathbb{R},$$

and let P_t^i be the semigroup corresponding to the process Y_t^i associated with $\mathcal{A}_i$ that spends zero time at 0. We let P_t be the semigroup corresponding to the process $Y_t = (Y_t^1, \ldots, Y_t^d)$, where the Y_t^i are independent. The independence implies that if $f(x) = \prod_{i=1}^d f^{(i)}(x_i)$ and $x = (x_1, \ldots, x_d)$, then $P_t f(x) = \prod_{i=1}^d P_t^i f^{(i)}(x_i)$.

We let U_t be the Poisson semigroup defined in terms of P_t:

$$U_t = \int_0^\infty \Big(\frac{t}{2\sqrt{\pi}} e^{-t^2/4s} s^{-3/2}\Big) P_s\,ds;$$

see [12], p. 127.

The semigroup P_t^i is self-adjoint on $(\mathbb{R}, \mu_i)$, where $\mu_i(dx) = |x_i|^{-\alpha_i}\,dx_i$. We let $\mu(dx) = \prod_{i=1}^d \mu_i(dx_i)$ be the product measure on $\mathbb{R}^d$.

Define

$$R_0 f(x) = \int_0^\infty P_t f(x)\, dt.$$

Lemma 6.1 *For f, h be bounded on $\mathbb{R}^d$ with compact support we have the identity*

$$\int (\mathcal{A}_i R_0 f(x)) h(x)\, \mu(dx) = \int \int_0^\infty y(\mathcal{A}_i U_{y/2} f(x))(U_{y/2} h(x))\, dy\, \mu(dx). \quad (6.1)$$

Proof. Using the spectral theorem, there exists (see [13]) a spectral representation

$$P_t^i = \int_0^\infty e^{-\lambda_i t} dE^i_{\lambda_i}, \qquad i = 1, \dots, d.$$

Write $s(\lambda) = \sum_{i=1}^d \lambda_i$ if $\lambda = (\lambda_1, \dots, \lambda_d)$. If $f(x) = \prod_{i=1}^d f^{(i)}(x_i)$, then

$$P_t f = \int_0^\infty \cdots \int_0^\infty e^{-ts(\lambda)} dE^1_{\lambda_1}(f^{(1)}) \cdots dE^d_{\lambda_d}(f^{(d)}),$$

and so

$$R_0 f = \int_0^\infty \cdots \int_0^\infty \frac{1}{s(\lambda)} dE^1_{\lambda_1}(f^{(1)}) \cdots dE^d_{\lambda_d}(f^{(d)}).$$

We have ([12], p. 127)

$$U_t f = \int_0^\infty \cdots \int_0^\infty e^{-t\sqrt{s(\lambda)}} dE^1_{\lambda_1}(f^{(1)}) \cdots dE^d_{\lambda_d}(f^{(d)}).$$

Note also

$$\mathcal{A}_i = \int_0^\infty \lambda_i dE^i_{\lambda_i}.$$

Therefore, if $h(x) = \prod_{i=1}^d h^{(i)}(x_i)$, the left hand side of (6.1) is

$$\int_0^\infty \cdots \int_0^\infty \frac{\lambda_i}{s(\lambda)} d(E^1_{\lambda_1}(f^{(1)}), E^1_{\lambda_1}(h^{(1)})) \cdots d(E^d_{\lambda_d}(f^{(d)}), E^d_{\lambda_d}(h^{(d)})). \quad (6.2)$$

We use here $(\,\cdot, \cdot\,)$ for the inner product in $L^2(\mu)$.

Similarly, the right hand side of (6.1) is

$$\int_0^\infty \Big[\int_0^\infty \cdots \int_0^\infty y\,\lambda_i e^{-(y/2)\sqrt{s(\lambda)}} e^{-(y/2)\sqrt{s(\lambda)}} d(E^1_{\lambda_1}(f^{(1)}), E^1_{\lambda_1}(h^{(1)}))$$
$$\cdots d(E^d_{\lambda_d}(f^{(d)}), E^d_{\lambda_d}(h^{(d)}))\Big]\,dy$$
$$= \int_0^\infty \Big[\int_0^\infty \cdots \int_0^\infty y\,\lambda_i e^{-y\sqrt{s(\lambda)}} d(E^1_{\lambda_1}(f^{(1)}), E^1_{\lambda_1}(h^{(1)}))$$
$$\cdots d(E^d_{\lambda_d}(f^{(d)}), E^d_{\lambda_d}(h^{(d)}))\Big]\,dy.$$

Since

$$\int_0^\infty y e^{-Ky}\,dy = \frac{1}{K^2},$$

this is equal to (6.2). Linear combinations of functions of the form $\prod_{i=1}^d f^{(i)}(x_i)$ are dense in $L^2(\mu)$, and an approximation argument completes the proof. □

Define for $(x,y) \in \mathbb{R}^d \times [0,\infty)$

$$G(f)(x) = \left(\int_0^\infty y\Big(\Big|\frac{\partial U_y f}{\partial y}(x,y)\Big|^2 + \sum_{i=1}^d |x_i|^{\alpha_i}\Big|\frac{\partial U_y f}{\partial x_i}(x,y)\Big|^2\Big)\,dy\right)^{1/2}. \quad (6.3)$$

The main result of [14] implies that if $1 < p < \infty$, then there exists c_p such that

$$\|G(f)\|_p \le c_p \|f\|_p, \quad (6.4)$$

where the norm is the $L^p(\mu)$ norm.

The main theorem of this section is the following.

Theorem 6.2 *Let $1 < p < \infty$. There exists a constant c_1 depending only on p such that*

$$\|\mathcal{A}_i R_0 f\|_p \le c_1 \|f\|_p.$$

Proof. Let f and h be bounded with compact support. Using Lemma 6.1,

integration by parts, Cauchy-Schwarz, and Hölder's inequality, we have

$$
\begin{aligned}
\int &(\mathcal{A}_i R_0 f(x)) h(x)\, \mu(dx) \\
&= \int \int_0^\infty y (\mathcal{A}_i U_{y/2} f(x))(U_{y/2} h(x))\, dy\, \mu(dx) \\
&= \int \int_0^\infty y |x_i|^{\alpha_i} \frac{\partial U_{y/2} f}{\partial x_i}(x) \frac{\partial U_{y/2} h}{\partial x_i}(x)\, dy\, \mu(dx) \\
&\le \int \Big(\int_0^\infty y |x_i|^{\alpha_i} \Big| \frac{\partial U_{y/2} f}{\partial x_i}(x) \Big|^2 dy \Big)^{1/2} \\
&\qquad \times \Big(\int_0^\infty y |x_i|^{\alpha_i} \Big| \frac{\partial U_{y/2} h}{\partial x_i}(x) \Big|^2 dy \Big)^{1/2} \mu(dx) \\
&\le \|G(f)\|_p \|G(h)\|_q,
\end{aligned}
$$

where q is the conjugate exponent to p. By (6.4) this in turn is bounded by

$$c_2 \|f\|_p \|h\|_q.$$

If we now take the supremum over all such h for which $\|h\|_q \le 1$, we obtain

$$\|\mathcal{A}_i R_0 f\|_p \le c_3 \|f\|_p$$

for f bounded with compact support. An approximation argument allows us to extend this inequality to all $f \in L^p$. □

Corollary 6.3 *Let $\lambda > 0$. Let $1 < p < \infty$. There exists a constant c_1 depending only on p such that*

$$\|\mathcal{A}_i R_\lambda f\|_p \le c_1 \|f\|_p.$$

Proof. If $f \in L^p$, then $f - \lambda R_\lambda f$ is also in L^p with $\|f - \lambda R_\lambda f\|_p \le 2\|f\|_p$. Our result now follows from Theorem 6.2 because $R_\lambda f = R_0(f - \lambda R_\lambda f)$. □

Remark 6.4 If $f \in L^p(\mathbb{R}^d_+, \mu)$, we can extend f to all of $\mathbb{R}^d$ by reflection. So Corollary 6.3 also applies when we look at the operators $\mathcal{A}_i$ and R_λ operating on functions whose domain is $\mathbb{R}^d_+$.

As with the first order estimates, only minor changes are needed if R_λ is the resolvent for $\sum_{i=1}^d a_i \mathcal{A}_i$, where the a_i are finite positive constants.

7 Uniqueness

We need one more estimate, and then we can complete the proof of Theorem 1.1.

Lemma 7.1 *If g is in C^2 with compact support contained in $(0,\infty)^d$, then for each $t > 0$ and $\lambda > 0$ we have that $P_t R_\lambda g$ is C^2 in $\mathbb{R}_+^d$ and for each i we have $(P_t R_\lambda g)_i = 0$ on Δ_i.*

Proof. We have a formula for the derivative of the transition density in the one-dimensional case in (8.6) below in the appendix. If we differentiate once more and use the fact that the transition densities for the process factor as a product of transition densities of one-dimensional processes, then tedious calculations show that $P_t g$ is C^2 with normal derivative 0 on the boundary. (This is somewhat easier than in the proof of Lemma 5.1 since we can use the fact that g has compact support.) Moreover one can show that the second derivatives of $P_t g$ grow with t at most polynomially. Since $P_t R_\lambda g = \int_0^\infty e^{-\lambda s} P_{s+t} g \, ds$ by the semigroup property, the lemma follows. □

The existence part of Theorem 1.1 was done in Section 4. It remains to prove uniqueness.

Proof of uniqueness in Theorem 1.1. Fix $x_0 \in \mathbb{R}_+^d$ and let $\varepsilon > 0$ be specified later. As in the nondegenerate case, to prove uniqueness it suffices to consider only the case where

$$\sum_{i=1}^d |a_i(y) - a_i(x_0)| \le \varepsilon, \qquad y \in \mathbb{R}_+^d; \tag{7.1}$$

see [4], Section VI.3. Let p_0 be the positive real given by Theorem 3.3. Set

$$\mathcal{L}_0 f(x) = \sum_{i=1}^d a_i(x_0) x_i^{\alpha_i} \frac{\partial^2 f}{\partial x_i^2}(x)$$

and let

$$\mathcal{B} = \mathcal{L} - \mathcal{L}_0.$$

Let R_λ and P_t be the resolvent and semigroup, respectively, for the operator $\mathcal{L}_0$. Taking into account Remarks 5.5 and 6.4, by Theorem 5.4 and Corollary 6.3 we have

$$\|\mathcal{B}R_\lambda f\|_{p_0} \le c_1\Big(d\varepsilon + \lambda^{\frac{1}{2b_i}-1}\sum_{i=1}^d \|b_i\|_\infty\Big)\|f\|_{p_0}. \tag{7.2}$$

Let us now choose ε small enough and λ large enough so that by (7.2) we have

$$\|\mathcal{B}R_\lambda f\|_{p_0} \le \tfrac{1}{2}\|f\|_{p_0}. \tag{7.3}$$

Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be any two solutions to the submartingale problem for $\mathcal{L}$ started at x_0, where we continue to assume (7.1) holds. We also assume that under each $\mathbb{P}_i$ the process spends zero time on Δ. Define

$$S^i_\lambda h = \mathbb{E}_i \int_0^\infty e^{-\lambda t} h(X_t)\,dt, \qquad i = 1, 2.$$

Let $R_K = \inf\{t : \sum_{i=1}^d L_t^{X^i} \ge K\}$. By Ito's formula, if $f \in C^2$ and $f_i = 0$ on Δ_i for each i, then

$$\mathbb{E}_i f(X_{t\wedge R_K}) - f(x_0) = \mathbb{E}_i \int_0^{t\wedge R_K} \mathcal{L}f(X_s)\,ds, \qquad i = 1, 2.$$

We let $K \to \infty$, so that $R_K \to \infty$. we then multiply both sides by $\lambda e^{-\lambda t}$ and integrate over t from 0 to ∞ to obtain

$$\lambda S^i_\lambda f - f(x_0) = S^i_\lambda \mathcal{L} f = S^i_\lambda \mathcal{L}_0 f + S^i_\lambda \mathcal{B} f. \tag{7.4}$$

Now let g be C^2 with compact support contained in $(0,\infty)^\infty$ and let $f = P_t R_\lambda g$. By Lemma 7.1 we can apply (7.4) to f. Note

$$\mathcal{L}_0 f = \mathcal{L}_0 R_\lambda P_t g = \lambda R_\lambda P_t g - P_t g.$$

Therefore (7.4) becomes

$$S^i_\lambda P_t g = R_\lambda P_t g + S^i_\lambda \mathcal{B} R_\lambda P_t g. \tag{7.5}$$

Let

$$\Theta = \sup_{\|h\|_{p_0}\le 1} |S^1_\lambda h - S^2_\lambda h|.$$

By Theorem 3.3 we know $\Theta < \infty$. By (7.5) and (7.3),

$$\begin{aligned}|S^1_\lambda P_t g - S^2_\lambda P_t g| &\le \Theta \|\mathcal{B} R_\lambda P_t g\|_{p_0}\\ &\le \tfrac{1}{2}\Theta \|P_t g\|_{p_0}\\ &\le \tfrac{1}{2}\Theta \|g\|_{p_0}.\end{aligned}$$

The last inequality follows by Jensen's inequality and the fact that P_t is self-adjoint with respect to μ. Since the support of g is disjoint from Δ, we can let $t \to 0$ and obtain

$$|S^1_\lambda g - S^2_\lambda g| \le \tfrac{1}{2}\Theta \|g\|_{p_0}.$$

We now take the supremum over all such g that in addition satisfy $\|g\|_{p_0} \le 1$. Since neither S^1_λ nor S^2_λ charge Δ, we then have

$$\Theta \le \tfrac{1}{2}\Theta.$$

Since $\Theta < \infty$, we conclude $\Theta = 0$.

From this point on we follow the proof of the nondegenerate case; see [4], Section VI.3. $\square$

8 Appendix

We prove Lemma 5.1.

We will use the well known facts (see [9], pp. 150–152):

$$I_p'(x) = I_{p+1}(x) + \frac{p}{x} I_p(x), \tag{8.1}$$

$$I_p'(x) = I_{p-1}(x) - \frac{p}{x} I_p(x), \tag{8.2}$$

$$I_p(x) \sim \frac{1}{2^p p!} x^p, \qquad x \to 0, \tag{8.3}$$

$$I_p(x) \sim \frac{1}{\sqrt{2\pi}} \frac{e^x}{\sqrt{x}}, \qquad x \to \infty. \tag{8.4}$$

In what follows we will take $p = \nu$ or $\nu + 1$ and $\nu = -1/(2b)$.

If we let $F(x) = I_{\nu+1}(x) - I_\nu(x)$, then from (8.1) and (8.2) we have

$$F'(x) = -F(x) - \frac{\nu+1}{x} I_{\nu+1}(x) - \frac{\nu}{x} I_\nu(x).$$

Using (8.4)

$$|F'(x) + F(x)| \le c_1 \frac{e^x}{x^{3/2}},$$

or

$$|(e^x F(x))'| \le c_1 \frac{e^{2x}}{x^{3/2}}.$$

Therefore

$$|e^x F(x)| \le |eF(1)| + c_1 \int_1^x \frac{e^{2y}}{y^{3/2}} dy.$$

By l'Hôpital's rule, the integral is bounded by $c_2 e^{2x}/x^{3/2}$, and so we deduce

$$|I_{\nu+1}(x) - I_\nu(x)| \le c_3 \frac{e^x}{x^{3/2}} \tag{8.5}$$

for $x \ge 1$.

Proof of Lemma 5.1. We start with (5.3). By scaling it suffices to do the case $t = 1$. Differentiating (5.1) we have

$$\frac{\partial p_X(1,x,y)}{\partial x} = cx^{b-\frac{1}{2}} y^{2b-\frac{3}{2}} e^{-K(x^{2b}+y^{2b})/2} [-x^b I_\nu(Kx^b y^b) + y^b I_{\nu+1}(Kx^b y^b)], \tag{8.6}$$

where K is some fixed positive constant. We write

$$\int_0^\infty \Big| \frac{\partial p_X(1,x,y)}{\partial x} \Big| dy = \int_0^{1/x} + \int_{1/x}^\infty := S_1 + S_2.$$

Using the bounds on I_ν,

$$S_1 \le cx^{2b-1} e^{-Kx^{2b}/2} \int_0^{1/x} [y^{2b-2} + y^{4b-2}] e^{-Ky^{2b}/2} dy.$$

Since $2b - 2 > -1$, the integral term is finite. Since $2b - 1 > 0$, the factor in front of the integral is bounded independently of x, so S_1 is bounded independently of x.

Since

$$\begin{aligned}
&| - x^b I_\nu(Kx^b y^b) + y^b I_{\nu+1}(Kx^b y^b)| \\
&\qquad\qquad = |(y^b - x^b) I_\nu(Kx^b y^b) + y^b (I_{\nu+1}(Kx^b y^b) - I_\nu(Kx^b y^b))| \\
&\qquad\qquad \le c|y^b - x^b| e^{Kx^b y^b} x^{-\frac{b}{2}} y^{-\frac{b}{2}} + y^b (e^{Kx^b y^b} x^{-b} y^{-b})
\end{aligned}$$

for $y \geq 1/x$, to bound S_2 we need to bound

$$\begin{aligned}\int_{1/x}^{\infty} &|y^b - x^b| x^{\frac{b-1}{2}} y^{\frac{3b-3}{2}} e^{-K(y^b-x^b)^2/2} dy \\ &\quad + \int_{1/x}^{\infty} x^{-\frac{1}{2}} y^{2b-\frac{3}{2}} e^{-K(y^b-x^b)^2/2} dy \\ &= S_3 + S_4.\end{aligned}$$

For S_3 we make the substitution $z = y^b - x^b$ and then

$$S_3 = c \int_{x^{-b}-x^b}^{\infty} |z| x^{\frac{b-1}{2}} (x^b + z)^{\frac{b-1}{2b}} c^{-kz^2/2} dz.$$

Since $(b-1)/(2b) < 0$ and $x^b + z \geq x^b$, this is less than

$$c \int_{-\infty}^{\infty} |z| x^{\frac{b-1}{2}} (x^{-b})^{\frac{b-1}{2b}} e^{-Kz^2/2} dz$$

which is bounded independently of x. For S_4 we use the same substitution. Since $2b - 1 > 0$, we have

$$(x^b + z)^{\frac{2b-1}{2b}} \leq c(x^{\frac{2b-1}{2}} + z^{\frac{2b-1}{2b}}).$$

Hence

$$\begin{aligned}S_4 &\leq \int_{x^{-b}-x^b}^{\infty} x^{-\frac{1}{2}} (x^b + z)^{\frac{2b-1}{2b}} e^{-Kz^2/2} dz \\ &\leq c \int_{x^{-b}-x^b}^{\infty} (x^{b-1} + x^{-\frac{1}{2}} z^{\frac{2b-1}{2b}}) e^{-Kz^2/2} dz.\end{aligned}$$

For each $p \geq 1$ there exists $c(p)$ such that

$$\int_a^{\infty} (1 + z)^{\frac{2b-1}{2b}} e^{-Kz^2/2} dz \leq c(p) a^{-p}, \qquad a > 1. \tag{8.7}$$

From this we see that S_4 is bounded independently of x for $x \leq 1$. On the other hand, for $x \geq 1$,

$$S_4 \leq c \int_{-\infty}^{\infty} (1 + |z|^{\frac{2b-1}{2b}}) e^{-Kz^2/2} dz \leq c.$$

We now turn to the proof of (5.4). Again by scaling we may assume $t = 1$. Looking at $\int_0^{1/y}$ and using the bounds on I_ν,

$$
\begin{aligned}
y^\alpha & \int_0^{1/y} \Big|\frac{\partial p_X(t,x,y)}{\partial x}\Big| x^{-\alpha} dx \\
& \le c\big[(y^{\alpha+2b-2} + y^{\alpha+4b-2}) e^{-Ky^{2b}/2}\big] \int_0^{1/y} x^{2b-1-\alpha} e^{-Kx^{2b}/2} dx.
\end{aligned}
$$

Since $\alpha + 2b - 2 = 0$, $\alpha + 4b - 2 > 0$, and $2b - 1 - \alpha = 1 - 2\alpha > -1$, the integral is finite and the expression in brackets is bounded in y.

To look at $\int_{1/y}^\infty$, we rewrite the integral as in S_2 and see that we have to bound

$$
\begin{aligned}
cy^{\frac{3}{2}b-\frac{3}{2}+\alpha} & \int_{1/y}^\infty x^{\frac{b}{2}-\frac{1}{2}-\alpha} |y^b - x^b| e^{-K(y^b-x^b)^2/2} dx \\
& \qquad + cy^{2b-\frac{3}{2}+\alpha} \int_{1/y}^\infty x^{-\frac{1}{2}-\alpha} e^{-K(y^b-x^b)^2/2} dx \\
& = S_5 + S_6.
\end{aligned}
$$

Letting $z = x^b - y^b$ as in S_3,

$$
\begin{aligned}
S_5 & \le cy^{\frac{3}{2}b-\frac{3}{2}+\alpha} \int_{y^{-b}-y^b}^\infty (y^b + z)^{-\frac{b+1+2\alpha}{2b}} z e^{-z^2/2} dx \\
& \le cy^{b-1} \int_{y^{-b}-y^b} z e^{-z^2/2} dz.
\end{aligned}
$$

When $y \le 1$ this is bounded using (8.7). When $y \ge 1$ this is bounded because $b - 1 < 0$.

For S_6 we have

$$
\begin{aligned}
S_6 & \le cy^{2b-\frac{3}{2}+\alpha} \int_{y^{-b}-y^b}^\infty (y^b + z)^{-\frac{b-3-2\alpha}{2b}} e^{-Kz^2/2} dz \\
& \le cy^{\frac{b}{2}-3} \int_{y^{-b}-v^b}^\infty e^{-Kz^2/2} dz.
\end{aligned}
$$

This is bounded in y for $y \le 1$ by (8.7). This is bounded in y for $y > 1$ because $\frac{b}{2} - 3$ is negative. $\square$

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